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Some modifications of Newton's method with fifth-order convergence

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Abstract

In this paper, we present some new modifications of Newton's method for solving non-linear equations. Analysis of convergence shows that these methods have order of convergence five. Numerical tests verifying the theory are given and based on these methods, a class of new multistep iterations is developed.
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1. Introduction

Solving non-linear equations is one of the most important problems in numerical analysis. In this paper, we consider iterative methods to find a simple root of a non-linear equation $f(x)=0$, where $f: D \subset \mathbb{R} \rightarrow \mathbb{R}$ for an open interval D is a scalar function.

The classical Newton's (CN) method for a single non-linear equation is written as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{1}$$

This is an important and basic method [9], which converges quadratically.

Some modifications of Newton's method with cubic convergence have been developed in [11,1,10,2], by considering different quadrature formulae for the computation of the integral arising from Newton's theorem

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(t)\,\mathrm{d}t. \tag{2}$$

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Weerakoon and Fernando [11] rederive the CN's method by the rectangular rule to compute the integral of (2) and by the trapezoidal approximation, they arrive at an implicit scheme

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_{n+1}) + f'(x_n)}, \tag{3}$$

which requires having the $(n+1)$th iterate x_{n+1} to calculate itself. They overcome this difficulty by making use of Newton's iterative step to compute the $(n+1)$th iterate on the right-hand side of (3). So a modified Newton's method with cubic convergence is obtained

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x^*_{n+1}) + f'(x_n)}, \tag{4}$$

where $x^*_{n+1} = x_n - f(x_n)/f'(x_n)$.

The midpoint rule for the integral of (2) gives that [1,10]

$$x_{n+1} = x_n - \frac{f(x_n)}{f'\left(\frac{1}{2}(x_n + x^*_{n+1})\right)}. \tag{5}$$

Scheme (5) has also been derived in [6] independently. The multivariate case is treated in [7,3].

In [8], instead of using the Newton's theorem for $y = f(x)$, Homeier uses it for the inverse function

$$x(y) = x(y_n) + \int_{y_n}^{y} x'(t)\,\mathrm{d}t,$$

to obtain a class of cubically convergent Newton-type methods, the best efficient one of which is

$$x_{n+1} = x_n - \frac{f(x_n)}{2}\left(\frac{1}{f'(x_n)} + \frac{1}{f'(x^*_{n+1})}\right). \tag{6}$$

Scheme (6) has also been derived in [10] independently.

On the other hand, Grau and Díaz-Barrero [5] propose an improvement of the Euler–Chebyshev method (see [5] and the references therein) with fifth-order convergence. This method is very interesting because it improves the order of convergence and computational efficiency of Euler–Chebyshev method with an additional evaluation of the function. In this paper, we will study such improvements of the above modifications of Newton's method.

Here, we present some new modifications of Newton's method with an additional evaluation of the function at another point iterated by the above modifications of Newton's method. These methods are proved to have the order of convergence five. Per iteration the new methods require two evaluations of the function and two of its first derivative. Their practical utility is demonstrated by numerical results.

2. The methods and analysis of convergence

Here, we express the third-order modifications of Newton's method mentioned in Section 1 as a general form

$$u_{n+1} = g_3(x_n). \tag{7}$$

Now, we consider the computation of the indefinite integral on a new interval of integration arising from Newton's theorem

$$f(x) = f(u_{n+1}) + \int_{u_{n+1}}^{x} f'(t)\,\mathrm{d}t. \tag{8}$$

By the rectangular rule to compute the integral of (8), we obtain

$$x_{n+1} = u_{n+1} - \frac{f(u_{n+1})}{f'(u_{n+1})}. \tag{9}$$

Similar to the derivation of modified Newton's methods, we approximate $f'(u_{n+1})$ of (9) with $f'(x^*_{n+1})$, where $x^*_{n+1} = x_n - f(x_n)/f'(x_n)$ is the Newton's iterate. So a class of new methods is

$$x_{n+1} = u_{n+1} - \frac{f(u_{n+1})}{f'(x^*_{n+1})}, \tag{10}$$

where u_{n+1} is defined by (7).

Moreover, using Taylor expansion, we have

$$f'(x^*_{n+1}) \simeq f'(x_n) + f''(x_n)(x^*_{n+1} - x_n), \tag{11}$$

$$f'\left(\tfrac{1}{2}(x_n + x^*_{n+1})\right) \simeq f'(x_n) + \tfrac{1}{2} f''(x_n)(x^*_{n+1} - x_n). \tag{12}$$

From (11) and (12), we can approximate

$$f'(x^*_{n+1}) \simeq 2f'(\tfrac{1}{2}(x_n + x^*_{n+1})) - f'(x_n). \tag{13}$$

Using (13) in (10), we can obtain the other class of new methods

$$x_{n+1} = u_{n+1} - \frac{f(u_{n+1})}{2f'\left(\tfrac{1}{2}(x_n + x^*_{n+1})\right) - f'(x_n)}, \tag{14}$$

where $x^*_{n+1} = x_n - f(x_n)/f'(x_n)$ and u_{n+1} is defined by (7). For (10) and (14), we have:

Theorem 1. *Assume that the function $f : D \subset \mathbb{R} \to \mathbb{R}$ for an open interval D has a simple root $\alpha \in D$. Let $f(x)$ has first, second and third derivatives in the interval D, then the methods defined by* (10) *and* (14), *in which u_{n+1} is defined by* (7) *and satisfies*

$$u_{n+1} - \alpha = Ae_n^3 + O(e_n^4), \tag{15}$$

for some $A \neq 0$, and $e_n = x_n - \alpha$, have the order of convergence five.

Proof. Using Taylor expansion and taking into account $f(\alpha) = 0$, we have

$$f(x_n) = f'(\alpha)[e_n + c_2 e_n^2 + c_3 e_n^3 + O(e_n^4)], \tag{16}$$

where $c_k = (1/k!) f^{(k)}(\alpha)/f'(\alpha)$, $k = 2, 3, \ldots$. Furthermore, we have

$$f'(x_n) = f'(\alpha)[1 + 2c_2 e_n + 3c_3 e_n^2 + O(e_n^3)]. \tag{17}$$

Dividing (16) by (17) gives us

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 + 2\left(c_2^2 - c_3\right) e_n^3 + O(e_n^4), \tag{18}$$

and hence, we have

$$x^*_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = \alpha + c_2 e_n^2 - 2(c_2^2 - c_3)e_n^3 + O(e_n^4). \tag{19}$$

Again expanding $f'(x^*_{n+1})$ about α and using (19), we have

$$f'(x^*_{n+1}) = f'(\alpha)[1 + 2c_2^2 e_n^2 + O(e_n^3)],$$

which reciprocal is

$$\frac{1}{f'(x^*_{n+1})} = \frac{1}{f'(\alpha)}[1 - 2c_2^2 e_n^2 + O(e_n^3)]. \tag{20}$$

Taylor expansion of $f(u_{n+1})$ is

$$f(u_{n+1}) = f'(\alpha)[(u_{n+1} - \alpha) + \mathrm{O}((u_{n+1} - \alpha)^2)]. \tag{21}$$

Since from (10) we have

$$e_{n+1} = u_{n+1} - \alpha - \frac{f(u_{n+1})}{f'(x^*_{n+1})},$$

from (15), (20) and (21), we have

$$\begin{aligned} e_{n+1} &= u_{n+1} - \alpha - [(u_{n+1} - \alpha) - 2c_2^2 e_n^2 (u_{n+1} - \alpha) + \mathrm{O}(e_n^6)] \\ &= 2c_2^2 e_n^2 (u_{n+1} - \alpha) + \mathrm{O}(e_n^6) \\ &= 2c_2^2 A e_n^5 + \mathrm{O}(e_n^6). \end{aligned} \tag{22}$$

This means that the methods defined by (10) are of fifth order.

In a similar way we can prove that the methods defined by (14) are of fifth order. □

From (10) and (14), we can obtain three new fifth-order modifications of Newton's method, in which the corresponding value u_{n+1} is defined by (4)–(6), respectively:

$$\begin{cases} u_{n+1} = x_n - \frac{2f(x_n)}{f'(x^*_{n+1}) + f'(x_n)}, \\ x_{n+1} = u_{n+1} - \frac{f(u_{n+1})}{f'(x^*_{n+1})}, \end{cases} \tag{23}$$

$$\begin{cases} u_{n+1} = x_n - \frac{f(x_n)}{f'\left(\frac{1}{2}(x_n + x^*_{n+1})\right)}, \\ x_{n+1} = u_{n+1} - \frac{f(u_{n+1})}{2f'\left(\frac{1}{2}(x_n + x^*_{n+1})\right) - f'(x_n)}, \end{cases} \tag{24}$$

and

$$\begin{cases} u_{n+1} = x_n - \dfrac{f(x_n)}{2} \left(\dfrac{1}{f'(x_n)} + \dfrac{1}{f'(x^*_{n+1})} \right), \\ x_{n+1} = u_{n+1} - \dfrac{f(u_{n+1})}{f'(x^*_{n+1})}, \end{cases} \tag{25}$$

where $x^*_{n+1} = x_n - f(x_n)/f'(x_n)$.

In fact, if each of three third-order methods defined by (4)–(6) is combined with either (10) or (14), we can obtain six new fifth-order modifications of Newton's method. However, compared with the methods defined by (23)–(25), the others require an additional evaluation of the first derivative and therefore they are less efficient.

From Theorem 1, using the error equations of the methods defined by (4)–(6) obtained in [11,10,8], respectively, it is easy to obtain that the method defined by (23) satisfies the following error equation:

$$e_{n+1} = c_2^2(2c_2^2 + c_3)e_n^5 + \mathrm{O}(e_n^6), \tag{26}$$

the method defined by (24) satisfies

$$e_{n+1} = \left(2c_2^2 - \tfrac{3}{2}c_3\right)\left(c_2^2 - \tfrac{1}{4}c_3\right) e_n^5 + \mathrm{O}(e_n^6), \tag{27}$$

and the method defined by (25) satisfies

$$e_{n+1} = c_2^2 c_3 e_n^5 + \mathrm{O}(e_n^6), \tag{28}$$

where $e_n = x_n - \alpha$ and $c_k = (1/k!)f^{(k)}(\alpha)/f'(\alpha)$, $k = 2, 3, \ldots$.

It is easy to know that per iteration the number of function evaluation (NFE) of the methods defined by (23)–(25) is four. We consider the definition of efficiency index [4] as $p^{1/w}$, where p is the order of the method and w is the NFEs per iteration required by the method. We have that the methods defined by (23)–(25) have the efficiency indexes equal to $\sqrt[4]{5} \simeq 1.495$, which are better than the ones of the cubically convergent methods $\sqrt[3]{3} \simeq 1.442$ and Newton's method $\sqrt{2} \simeq 1.414$.

3. Numerical examples

After the nomenclature used in [10], iterative formulae (1), (4)–(6) are, respectively, called the CN's method, arithmetic mean Newton's (AN) method, midpoint Newton's (MN) method and harmonic mean Newton's (HN) method. Here, we use the logograms as FAN, FMN and FHN to represent the present fifth-order methods defined by (23)–(25), respectively. The performance of the present methods with CN, AN, MN and HN is compared. Displayed in Table 1 is the NFEs required such that $|f(x_n)| < 1.\mathrm{E} - 14$.

The results in Table 1 show that the present methods improve the computational efficiency of CN and its known modifications, namely AN, MN and HN. As far as the results we consider, FHN requires the less NFEs compared to various methods. Moreover, the present methods can compete with CN.

We use the following functions, most of which are the same as in [11,5], respectively:

$$f_1(x) = x^3 + 4x^2 - 10, \quad \alpha = 1.3652300134140969,$$

$$f_2(x) = x^2 - \mathrm{e}^x - 3x + 2, \quad \alpha = 0.25753028543986084,$$

$$f_3(x) = x\mathrm{e}^{x^2} - \sin^2(x) + 3\cos(x) + 5, \quad \alpha = -1.207647827130919,$$

$$f_4(x) = \sin(x)\mathrm{e}^{-x} + \ln(x^2 + 1), \quad \alpha = 0,$$

$$f_5(x) = (x-1)^3 - 1, \quad \alpha = 2,$$

$$f_6(x) = \cos(x) - x, \quad \alpha = 0.73908513321516067,$$

$$f_7(x) = x^2 + \sin(x/5) - 1/4, \quad \alpha = 0.4099920179891371,$$

$$f_8(x) = \mathrm{e}^x - 4x^2, \quad \alpha = 0.7148059123627778.$$

Table 1
Comparison of various iterative methods

	x_0	CN	AN	MN	HN	FAN	FMN	FHN
f_1	−0.3	108	18	54	207	16	44	84
	1	10	9	9	9	12	8	8
f_2	1	8	9	9	9	8	8	8
	2	10	12	9	12	8	12	8
f_3	−0.5	20	33	15	12	32	16	12
	−1.5	12	12	12	12	12	12	12
f_4	1.5	10	12	12	9	8	12	8
f_5	1.5	14	15	12	12	16	12	12
	3.5	14	15	15	12	12	12	12
f_6	0.2	10	9	9	9	8	8	8
	1.7	8	9	9	9	8	8	8
f_7	0.3	8	9	9	6	8	8	8
	0.7	10	9	9	9	8	8	8
f_8	0	14	12	12	12	12	12	12
	1	10	9	9	9	8	8	8

4. Further developments

The well-known multistep Newton's method may be expressed as

$$
\begin{aligned}
&x_n^{(0)} = x_n,\\
&x_n^{(i)} = x_n^{(i-1)} - f'(x_n)^{-1} f(x_n^{(i-1)}), \quad i = 1, \ldots, m \text{ and } m \geqslant 1,\\
&x_{n+1} = x_n^{(m)}.
\end{aligned} \tag{29}
$$

The classical Newton's method may be viewed as the particular case for $m = 1$ in (29). And also we know that this scheme is of order $(m + 1)$.

Here, we consider a class of new multistep iterations

$$
\begin{aligned}
&x_{n+1}^{(0)} = u_{n+1},\\
&x_{n+1}^{(i)} = x_{n+1}^{(i-1)} - \mu f(x_{n+1}^{(i-1)}), \quad i = 1, \ldots, m \text{ and } m \geqslant 1,\\
&x_{n+1} = x_{n+1}^{(m)},
\end{aligned} \tag{30}
$$

where u_{n+1} is defined by one of the methods defined by (7) and the corresponding μ is defined by $\mu = f'(x_{n+1}^*)^{-1}$ or $\mu = [2f'(\frac{1}{2}(x_n + x_{n+1}^*)) - f'(x_n)]^{-1}$. When we take $m = 1$, the iterations (30) become (23)–(25). For (30), we have:

Theorem 2. *Under the hypotheses of Theorem* 1, *the multistep iterations defined by* (30) *are of order* $(2m + 3)$.

Proof. Let $e_n = x_n - \alpha$ and $e_{n+1}^{(i)} = x_{n+1}^{(i)} - \alpha$. For the case $\mu = f'(x_{n+1}^*)^{-1}$, from (20) and Taylor expansion of $f(x_{n+1}^{(i-1)})$ about α, we have

$$
e_{n+1}^{(i)} = 2c_2^2 e_n^2 e_{n+1}^{(i-1)} + \mathrm{O}(e_n^3 e_{n+1}^{(i-1)}), \quad 1 \leqslant i \leqslant m,
$$

which yields the desired results.

In a similar way, for the case $\mu = [2f'(\frac{1}{2}(x_n + x_{n+1}^*)) - f'(x_n)]^{-1}$, we can prove that the methods defined by (30) are of order $(2m + 3)$. □

When adding m evaluations of the function, the order of the iterations (30) adds $2m$, while the order of multistep Newton method only adds m. This means that the new iterations (30) are superior than multistep Newton's method. However, the efficiency index of this class of new multistep iterations is equal to $(2m + 3)^{1/(m+3)}$, which is smaller than the ones of the methods defined by (23)–(25) ($\sqrt[4]{5}$) if $m > 1$. Thus, in the efficiency, the multistep iterations (30) ($m > 1$) have no advantages over the methods defined by (23)–(25), which have already proved to be very efficient.

5. Conclusions

We have obtained some new modifications of Newton's method. From Theorem 1, we prove that the methods have the order of convergence five. Analysis of efficiency shows that these methods can compete with Newton's method, which is also demonstrated by numerical results. Also based on these methods, we develop a class of new multistep iterations, which is better than multistep Newton's method.

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